

CAŁKI NIEOZNACZONE.

1. Obliczyć całki: a) $\int \frac{(x + \sqrt{x})^2}{x^3} dx$ b) $\int \frac{x^2 + 2x^2\sqrt{x} + 2}{x^3} dx$ c) $\int \frac{\sqrt{x} - 2}{x\sqrt{x}} dx$ d) $\int \frac{2x+1}{x^2+1} dx$ e) $\int \frac{3x-1}{x^2+1} dx$.

2. Całkując przez części, obliczyć: a) $\int \arctg x dx$ b) $\int x^5 \ln x dx$ c) $\int \frac{\ln x}{x^3} dx$ d) $\int \ln x dx$ e) $\int \ln^2 x dx$
 f) $\int x^2 \arctg x dx$ g) $\int x \ln(x^2 + 1) dx$.

3. Używając podstawień, obliczyć całki: a) $\int \frac{dx}{e^x + e^{-x}}$ b) $\int x \cdot \sqrt{4-x^2} dx$ c) $\int \frac{\cos x}{1+4\sin^2 x} dx$ d) $\int e^{\sqrt{x}} dx$

e) $\int \frac{e^x}{x^3} dx$ f) $\int \frac{\operatorname{ctg} x}{\ln(\sin x)} dx$ g) $\int \frac{\cos x}{\sqrt{5+3\sin x}} dx$ h) $\int \frac{x dx}{\sqrt{1-x^4}}$ i) $\int \frac{\operatorname{tg} x}{1+\operatorname{tg}^4 x} \cdot \frac{dx}{\cos^2 x}$ j) $\int \frac{\ln \operatorname{tg} x}{\sin x \cdot \cos x} dx$.

d) $\int \frac{x+3}{\sqrt{2+x}} dx$ e) $\int \frac{\sin^3 x}{\cos^2 x} dx$ f) $\int \frac{\cos x}{\sqrt{1+\sin x}} dx$ g) $\int \frac{\cos x}{\sin^4 x} dx$ j) $\int \frac{dx}{x\sqrt{2+\ln x}}$ k) $\int \frac{\sqrt[3]{\operatorname{tg} x + 2}}{\cos^2 x} dx$

4. Znaleźć całki funkcji wymiernych: a) $\int \frac{dx}{4x^2+9}$ b) $\int \frac{x+5}{x^2+4x+8} dx$ c) $\int \frac{dx}{x^2-8x+20}$

d) $\int \frac{3x+1}{x^2+6x+13} dx$ e) $\int \frac{x dx}{x^2+2x-3}$ f) $\int \frac{x^2+7x}{x^2+5x-6} dx$ g) $\int \frac{x^2+6}{x^2+3x-4} dx$

5. Obliczyć całki funkcji trygonometrycznych: a) $\int \frac{\sin^3 x}{\cos^2 x} dx$ b) $\int \sin^6 x \cos^5 x dx$ c) $\int \sin^5 x dx$

d) $\int \frac{\cos 2x}{\cos^2 x \cdot \sin^2 x} dx$ e) $\int \operatorname{tg}^3 x dx$ f) $\int \frac{dx}{\sin x \cos x}$.

ROZWIĄZANIA NIEKTÓRYCH ZADAŃ.

1. a) $\int \frac{(x + \sqrt{x})^2}{x^3} dx = \int \frac{x^2 + 2x\sqrt{x} + x}{x^3} dx = \int \frac{1}{x} dx + 2 \int x^{-\frac{3}{2}} dx + \int x^{-2} dx = \ln|x| - 4x^{-\frac{1}{2}} - x^{-1} = \ln|x| - \frac{4}{\sqrt{x}} - \frac{1}{x^2}$.

d) $\int \frac{2x+1}{x^2+1} dx = \int \frac{2x}{x^2+1} dx + \int \frac{1}{x^2+1} dx = \ln(x^2+1) + \arctg x$.

2. a) $\int \arctg x dx = \left| \begin{array}{l} u = \arctg x \quad u' = \frac{1}{x^2+1} \\ v' = 1 \quad v = x \end{array} \right| = x \arctg x - \int \frac{x}{x^2+1} dx = x \arctg x - \frac{1}{2} \ln(x^2+1)$.

c) $\int \frac{\ln x}{x^3} dx = \left| \begin{array}{l} u = \ln x \quad u' = \frac{1}{x} \\ v' = \frac{1}{x^3} \quad v = -\frac{1}{2x^2} \end{array} \right| = -\frac{1}{2x^2} \ln x + \frac{1}{2} \int \frac{1}{x^3} dx = -\frac{1}{2x^2} \ln x + \frac{1}{2} \left(-\frac{1}{2x^2}\right) = -\frac{1}{2x^2} \ln x - \frac{1}{4x^2}$.

e) $\int \ln^2 x dx = \left| \begin{array}{l} u = \ln^2 x \quad u' = \frac{1}{x} \cdot 2 \ln x \\ v' = 1 \quad v = x \end{array} \right| = x \ln^2 x - 2 \int \ln x dx = \left| \begin{array}{l} u = \ln x \quad u' = \frac{1}{x} \\ v' = 1 \quad v = x \end{array} \right| = x \ln^2 x - 2(x \ln x - \int dx)$
 $= x \ln^2 x - 2x \ln x + 2x$.

2. g) $\int x \ln(x^2+1) dx = \left| \begin{array}{l} u = \ln(x^2+1) \quad u' = \frac{2x}{x^2+1} \\ v' = x \quad v = \frac{x^2}{2} \end{array} \right| = \frac{x^2}{2} \ln(x^2+1) - \int \frac{x^3}{x^2+1} dx =$
 $= \frac{x^2}{2} \ln(x^2+1) - \int \frac{x^3+x-x}{x^2+1} dx = \frac{x^2}{2} \ln(x^2+1) - \int \frac{x^3+x}{x^2+1} dx + \int \frac{x}{x^2+1} dx = \frac{x^2}{2} \ln(x^2+1) - \frac{x^2}{2} + \frac{1}{2} \ln(x^2+1)$.

$$3. a) \int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x dx}{e^{2x} + 1} \left[\begin{array}{l} t = e^x \\ dt = e^x dx \end{array} \right] = \int \frac{dt}{t^2 + 1} = \operatorname{arctg} t = \operatorname{arctg} e^x .$$

$$d) \int e^{\sqrt{x}} dx = \left[\begin{array}{l} t = \sqrt{x} \\ t^2 = x \\ 2t dt = dx \end{array} \right] = \int e^t 2t dt = \left[\begin{array}{l} u = 2t \quad u' = 2 \\ v' = e^t \quad v = e^t \end{array} \right] = 2t e^t - \int 2e^t dt = 2t e^t - 2e^t = 2\sqrt{x} \cdot e^{\sqrt{x}} - 2e^{\sqrt{x}} .$$

$$e) \int \frac{e^{\frac{1}{x}}}{x^3} dx = \left[\begin{array}{l} \frac{1}{x} = t \\ \frac{1}{x^2} dx = -dt \end{array} \right] = \int t \cdot e^t dt = \text{dalej przez części jak w d).}$$

$$f) \int \frac{\operatorname{ctg} x}{\ln(\sin x)} dx = \left[\begin{array}{l} t = \ln(\sin x) \\ dt = \frac{1}{\sin x} \cdot \cos x dx = \operatorname{ctg} x dx \end{array} \right] = \int \frac{dt}{t} = \ln t = \ln(\ln(\sin x)) .$$

$$h) \int \frac{x dx}{\sqrt{1-x^4}} = \left[\begin{array}{l} x^2 = t \\ x dx = \frac{1}{2} dt \end{array} \right] = \frac{1}{2} \arcsin t = \frac{1}{2} \arcsin x^2$$

$$k) \int \frac{\sqrt[3]{\operatorname{tg} x + 2}}{\cos^2 x} dx = \left[\begin{array}{l} t = \sqrt[3]{\operatorname{tg} x + 2} \\ t^3 = \operatorname{tg} x + 2 \\ 3t^2 dt = \frac{dx}{\cos^2 x} \end{array} \right] = \int t \cdot 3t^2 dt = 3 \int t^3 dt = \frac{3}{4} t^4 = \frac{3}{4} \sqrt[3]{(\operatorname{tg} x + 2)^4} .$$

4. W przykładach a) - d) jest $\Delta < 0$. a) $\int \frac{dx}{4x^2 + 9} = \frac{1}{9} \int \frac{dx}{(\frac{2}{3}x)^2 + 1} = \frac{1}{6} \operatorname{arctg}(\frac{2}{3}x)$.

$$b) \int \frac{x+5}{x^2+4x+8} dx = \int \frac{x+2}{x^2+4x+8} dx + \int \frac{3}{x^2+4x+8} dx = \frac{1}{2} \int \frac{2x+4}{x^2+4x+8} dx + \int \frac{3}{(\frac{x+2}{2})^2+1} dx =$$

$$= \ln(x^2+4x+8) + \frac{1}{2} \operatorname{arctg}(\frac{x+2}{2}) .$$

$$d) \int \frac{3x+1}{x^2+6x+13} dx = \int \frac{3x}{x^2+6x+13} dx + \int \frac{1}{x^2+6x+13} = \frac{3}{2} \int \frac{2x+6-6}{x^2+6x+13} dx + \int \frac{1}{x^2+6x+13} dx =$$

$$\frac{3}{2} \int \frac{2x+6}{x^2+6x+13} dx - 8 \int \frac{1}{(x+3)^2+4} dx = \ln(x^2+6x+13) - 8 \operatorname{arctg}(\frac{x+3}{2}) .$$

$$e) \int \frac{x dx}{x^2+2x-3} : \Delta > 0, x^2+2x-3 = (x-1)(x+3), \int \frac{x dx}{x^2+2x-3} = \frac{1}{4} \int \frac{dx}{x-1} + \frac{3}{4} \int \frac{dx}{x+3} = \frac{1}{4} \ln|x-1| + \frac{3}{4} \ln|x+3| .$$

$$g) \int \frac{x^2+6}{x^2+3x-4} dx = \int \frac{x^2+3x-4+10}{x^2+3x-4} dx = \int \frac{x^2+3x-4}{x^2+3x-4} dx + \int \frac{10}{x^2+3x-4} dx = \int dx + \int \frac{2}{x-1} dx - \int \frac{2}{x+4} dx =$$

$$= x + 2 \ln|x-1| - 2 \ln|x+4|$$

$$5. c) \int \sin^5 x dx = \int (\sin^2 x)^2 \sin x dx = \int (1 - \cos^2 x)^2 \sin x dx = \left[\begin{array}{l} \cos x = t \\ \sin x dx = -dt \end{array} \right] = - \int (1-t^2)^2 dt =$$

$$= - \int dt + 2 \int t^2 dt - \int t^4 dt = -t + \frac{2}{3} t^3 - \frac{1}{5} t^5 = -\sin x + \frac{2}{3} \sin^3 x - \frac{1}{5} \sin^5 x .$$

$$d) \int \frac{\cos 2x}{\cos^2 x \cdot \sin^2 x} dx = \int \frac{\cos^2 x - \sin^2 x}{\cos^2 x \cdot \sin^2 x} dx = \int \frac{1}{\sin^2 x} dx - \int \frac{1}{\cos^2 x} dx = -\operatorname{ctg} x - \operatorname{tg} x .$$

$$e) \int \operatorname{tg}^3 x dx = \int \frac{\sin^3 x}{\cos^3 x} dx = \int \frac{1 - \cos^2 x}{\cos^3 x} \sin x dx = \left[\begin{array}{l} \cos x = t \\ \sin x dx = -dt \end{array} \right] = \int \frac{1-t^2}{t^3} dt = \int t^{-3} dt + \int \frac{dt}{t} = \frac{t^{-2}}{-2} - \ln|t| =$$

$$= -\frac{1}{2 \cos^2 x} - \ln|\cos x|$$

$$f) \int \frac{dx}{\sin x \cos x} = \int \frac{\cos x dx}{\sin x \cos^2 x} = \int \operatorname{ctg} x \cdot \frac{dx}{\cos^2 x} = \int \frac{1}{\operatorname{tg} x} \cdot \frac{dx}{\cos^2 x} = \left[\begin{array}{l} t = \operatorname{tg} x \\ dt = \frac{dx}{\cos^2 x} \end{array} \right] = \int \frac{1}{t} dt = \ln t = \ln(\operatorname{tg} x) .$$