

Całki oznaczone, całki niewłaściwe, pole figury.

1. Znaleźć całki (w [...] są wyniki): a) $\int_{-4}^{-2} \frac{dx}{x^2 + 6x + 10}$ $\left[\frac{\pi}{2} \right]$ b) $\int_0^{\frac{\pi}{4}} \operatorname{tg} x \, dx$ $\left[\ln \sqrt{2} \right]$ c) $\int_0^{\frac{\pi}{2}} \sin^3 x \, dx$ $\left[\frac{2}{3} \right]$

d) $\int_0^{\frac{\pi}{2}} \sin^4 x \cdot \cos^3 x \, dx$ $\left[\frac{2}{35} \right]$ e) $\int_0^{\frac{\pi}{3}} \operatorname{tg}^3 x \, dx$ $\left[\frac{2}{3} - \ln 2 \right]$ f) $\int_0^{\frac{\sqrt{\pi}}{2}} x \sin x^2 \, dx$ $\left[\frac{1}{2} \right]$ g) $\int_0^{\frac{\pi}{3}} \cos^2 x \, dx$ $\left[\frac{\sqrt{3}}{8} + \frac{\pi}{6} \right]$

h) $\int_0^{\sqrt{3}} \frac{x \, dx}{\sqrt{x^2 + 1}}$ [1] i) $\int_0^5 \frac{x+2}{\sqrt{x+4}} \, dx$ $\left[\frac{26}{3} \right]$ j) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{2 + \sin x}} \, dx$ $\left[2\sqrt{3} - 2\sqrt{2} \right]$ k) $\int_{\sqrt{2}}^{\sqrt{5}} \frac{x}{\sqrt{x^2 - 1}} \, dx$ [1].

2. Obliczyć (lub wykazać rozbieżność) całki niewłaściwe: a) $\int_{-\infty}^{+\infty} \frac{2x \, dx}{x^2 + 1}$ [rozbieżna] b) $\int_0^{\infty} \frac{dx}{x^2 - 4x + 8}$ $\left[\frac{3\pi}{4} \right]$

c) $\int_1^{\infty} \frac{4x + \sqrt{x}}{x^2} \, dx$ [rozbież.] d) $\int_1^{\infty} \frac{4x + \sqrt{x}}{x^3} \, dx$ $\left[\frac{10}{3} \right]$ e) $\int_{-1}^0 \frac{e^{-\frac{1}{x}}}{x^3} \, dx$ [rozbież.] f) $\int_0^e \frac{dx}{x \ln^2 x}$ [1] g) $\int_0^4 \frac{4x}{x^2 \sqrt{x}} \, dx$ [rozbież.]

h) $\int_0^1 \frac{\sqrt{x}}{3x} \, dx$ $\left[\frac{2}{3} \right]$ i) $\int_0^2 \frac{dx}{(x-1)^2}$ j) $\int_{-1}^1 \frac{dx}{\sqrt[3]{x^5}}$ k) $\int_0^2 \frac{\sqrt{x}}{4x^3} \, dx$ l) $\int_{-1}^2 \frac{1}{x^2 - 2x - 3} \, dx$ [i, j, k, l - całki rozbieżne].

3. Obliczyć pole obszaru ograniczonego krzywymi (wykonać rysunek): a) $y = x - x^2, y = x^2 - 1$ $\left[\frac{9}{8} \right]$

b) $y = x^2 + x - 2, y = -x + 1$ $\left[\frac{32}{3} \right]$ c) $y = x^2 - 6x + 10, y = 6x - x^2$ $\left[21\frac{1}{3} \right]$ d) $y = \frac{2}{x}, y = x - 1, x = 4$ $\left[4 + \ln \frac{1}{4} \right]$

e) $y = 2 - x, y = \sqrt{x}, x = 0$ $\left[\frac{7}{6} \right]$ f) $y = \frac{x^2}{4}, y = \frac{8}{x^2 + 4}$ $\left[2\pi - \frac{4}{3} \right]$.

Rozwiązania niektórych zadań.

1. e) $\int_0^{\frac{\pi}{3}} \operatorname{tg}^3 x \, dx = \begin{bmatrix} \operatorname{tg} x = t \\ x = \operatorname{arctg} t \\ dx = \frac{dt}{t^2 + 1} \end{bmatrix} = \int_0^{\sqrt{3}} \frac{t^3}{1+t^2} \, dt = \int_0^{\sqrt{3}} \frac{t^3 + t - t}{1+t^2} \, dt = \int_0^{\sqrt{3}} \frac{t^3 + t}{1+t^2} \, dt - \frac{1}{2} \int_0^{\sqrt{3}} \frac{2t}{t^2 + 1} \, dt = \int_0^{\sqrt{3}} t \, dt - \frac{1}{2} \ln(t^2 + 1) \Big|_0^{\sqrt{3}} =$

x	0	$\frac{\pi}{3}$
t	0	$\sqrt{3}$

$$= \frac{t^2}{2} \Big|_0^{\sqrt{3}} - \frac{1}{2} \ln 4 - \frac{1}{2} \ln 1 = \frac{3}{2} - \ln 2.$$

f) $\int_0^{\frac{\sqrt{\pi}}{2}} x \sin x^2 \, dx = \begin{bmatrix} x^2 = t \\ 2x \, dx = dt \end{bmatrix} = \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin t \, dt = -\frac{1}{2} \cos t \Big|_0^{\frac{\pi}{2}} = 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}.$

x	0	$\frac{\sqrt{\pi}}{2}$
t	0	$\frac{\pi}{2}$

g) $\int_0^{\frac{\pi}{3}} \cos^2 x \, dx = \left[\frac{1 + \cos 2x}{2} \right] = \frac{1}{2} \int_0^{\frac{\pi}{3}} (1 + \cos 2x) \, dx = \frac{1}{2} x \Big|_0^{\frac{\pi}{3}} + \frac{1}{4} \sin 2x \Big|_0^{\frac{\pi}{3}} = \frac{\pi}{6} + \frac{1}{4} \sin \frac{2\pi}{3} = \frac{\pi}{6} + \frac{1}{4} \cdot \frac{\sqrt{3}}{2} = \frac{\pi}{6} + \frac{\sqrt{3}}{8}.$

$$h) \int_0^{\sqrt{3}} \frac{x dx}{\sqrt{x^2+1}} = \left[\begin{array}{l} \sqrt{x^2+1} = t \\ x^2+1 = t^2 \\ x dx = t dt \end{array} \right] = \int_1^2 \frac{t dt}{t} = t \Big|_1^2 = 2-1=1.$$

$$j) \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{2+\sin x}} dx = \left[\begin{array}{l} t = \sqrt{2+\sin x} \\ t^2 = 2+\sin x \\ 2t dt = \cos x dx \end{array} \right] = \int_{\sqrt{2}}^{\sqrt{3}} \frac{2t dt}{t} = 2t \Big|_{\sqrt{2}}^{\sqrt{3}} = 2\sqrt{3} - 2\sqrt{2}.$$

x	0	$\frac{\pi}{2}$
t	$\sqrt{2}$	$\sqrt{3}$

$$2.a) \int_{-\infty}^{+\infty} \frac{2x dx}{x^2+1} = \int_{-\infty}^0 \frac{2x dx}{x^2+1} + \int_0^{+\infty} \frac{2x dx}{x^2+1} = \lim_{a \rightarrow -\infty} \int_a^0 \frac{2x dx}{x^2+1} + \lim_{b \rightarrow +\infty} \int_0^b \frac{2x dx}{x^2+1} = \lim_{a \rightarrow -\infty} \ln(x^2+1) + \lim_{b \rightarrow +\infty} \ln(x^2+1) = \infty + \infty,$$

więc całka rozbieżna.

$$c) \int_1^{\infty} \frac{4x + \sqrt{x}}{x^2} dx = \lim_{a \rightarrow \infty} \int_1^a \frac{4x + \sqrt{x}}{x^2} dx = \lim_{a \rightarrow \infty} \left(4 \ln x - \frac{2}{\sqrt{x}} \right) \Big|_1^a = \lim_{a \rightarrow \infty} \left(4 \ln a - \frac{2}{a} - 4 \ln 1 + 2 \right) = \infty, \text{ więc całka rozbieżna.}$$

$$d) \int_1^{\infty} \frac{4x + \sqrt{x}}{x^3} dx = \lim_{a \rightarrow \infty} \int_1^a \frac{4x + \sqrt{x}}{x^3} dx = \lim_{a \rightarrow \infty} \left(-\frac{4}{x} - \frac{2}{3\sqrt{x^3}} \right) \Big|_1^a = \lim_{a \rightarrow \infty} \left(-\frac{4}{a} - \frac{2}{3\sqrt{a^3}} + 4 + \frac{2}{3} \right) = \frac{10}{3}.$$

$$f) \int_0^{\frac{1}{e}} \frac{dx}{x \ln^2 x} = \lim_{a \rightarrow 0^+} \left(-\frac{1}{\ln x} \right) \Big|_a^{\frac{1}{e}} = \lim_{a \rightarrow 0^+} \left(-\frac{1}{\ln e^{-1}} + \frac{1}{\ln a} \right) = -(-1) + 0 = 1$$

$$\int \frac{dx}{x \ln^2 x} = \left[\begin{array}{l} \ln x = t \\ \frac{1}{x} dx = dt \end{array} \right] = \int \frac{dt}{t^2} = -\frac{1}{t} = -\frac{1}{\ln x}.$$

$$i) \int_0^2 \frac{dx}{(x-1)^2} = \int_0^1 \frac{dx}{(x-1)^2} + \int_1^2 \frac{dx}{(x-1)^2} = \lim_{a \rightarrow 1^-} \int_0^a \frac{dx}{(x-1)^2} + \lim_{a \rightarrow 1^+} \int_a^2 \frac{dx}{(x-1)^2} = \lim_{a \rightarrow 1^-} \left(-\frac{1}{x-1} \Big|_0^a \right) + \lim_{a \rightarrow 1^+} \left(-\frac{1}{x-1} \Big|_a^2 \right) = \lim_{a \rightarrow 1^-} \left(-\frac{1}{a-1} - 1 \right) + \lim_{a \rightarrow 1^+} \left(1 + \frac{1}{a-1} \right) = \infty + \infty = \infty \text{ całka rozbieżna.}$$

$$l) \int_{-1}^2 \frac{1}{x^2-2x-3} dx = \lim_{a \rightarrow -1^+} \left(-\frac{1}{4} \ln|x+1| + \frac{1}{4} \ln|x-3| \right) \Big|_{-1}^a = \lim_{a \rightarrow -1^+} \left(-\frac{1}{4} \ln 3 + \frac{1}{4} \ln|2-3| + \frac{1}{4} \ln(a+1) - \frac{1}{4} \ln|a-3| \right) = \frac{1}{4} \ln \left(\frac{4}{3} \right) - \infty = -\infty \text{ całka rozbieżna.}$$

$$\frac{1}{x^2-2x-3} = \frac{1}{(x+1)(x-3)} = \frac{-\frac{1}{4}}{x+1} + \frac{\frac{1}{4}}{x-3}; \int \frac{1}{x^2-2x-3} dx = -\frac{1}{4} \ln|x+1| + \frac{1}{4} \ln|x-3|.$$

$$3c) y = x^2 - 6x + 10, y = 6x - x^2 \quad x^2 - 6x + 10 = 6x - x^2 \Leftrightarrow 2x^2 - 12x + 10 = 0 \Leftrightarrow x = 1 \text{ lub } x = 5.$$

$$P = \int_1^5 (6x - x^2 - x^2 + 6x - 10) dx = \int_1^5 (12x - 2x^2 - 10) dx = (6x^2 - \frac{2}{3}x^3 - 10x) \Big|_1^5 = 150 - \frac{250}{3} - 50 - 6 + \frac{2}{3} + 10 = 104 - \frac{248}{3} = 21\frac{1}{3}$$

$$d) y = \frac{2}{x}, y = x-1, x = 4 \quad \frac{2}{x} = x-1 \Leftrightarrow x^2 - x - 2 = 0 \Leftrightarrow x_1 = -1, x_2 = 2, P = \int_2^4 \left(x-1 - \frac{2}{x} \right) dx = 4 - 2 \ln 2.$$

$$g) y = \frac{x^2}{4}, y = \frac{8}{x^2+4} \text{ punkty wspólne dla } x = -2 \text{ lub } x = 2.$$

$$P = \int_{-2}^2 \left(\frac{8}{x^2+4} - \frac{x^2}{4} \right) dx = 4 \arctg \left(\frac{x}{2} \right) \Big|_{-2}^2 - \frac{x^3}{12} \Big|_{-2}^2 = 4 \arctg 1 - 4 \arctg(-1) - \frac{2}{3} - \frac{2}{3} = 2\pi - \frac{4}{3}.$$