Całki oznaczone, całki niewłaściwe, pole figury.

1. Znaleźć całki (w [.] są wyniki): a) $\int_{-4}^{-2}\frac{dx}{x^2+6x+10}$ $\left[\frac{\pi}{2}\right]$ b) $\int_0^{\frac{\pi}{4}}\operatorname{tg}x\,dx$ $\left[\ln\sqrt{2}\right]$ c) $\int_0^{\frac{\pi}{2}}\sin^3x\,dx$ $\left[\frac{2}{3}\right]$

d) $\int_0^{\frac{\pi}{2}}\sin^4x\cdot\cos^3x\,dx$ $\left[\frac{2}{35}\right]$ e) $\int_0^{\frac{\pi}{3}}\operatorname{tg}^3x\,dx$ $\left[\frac{2}{3}-\ln 2\right]$ f) $\int_0^{\sqrt{\frac{\pi}{2}}}x\sin x^2dx$ $\left[\frac{1}{2}\right]$ g) $\int_0^{\frac{\pi}{3}}\cos^2x\,dx$ $\left[\frac{\sqrt{3}}{8}+\frac{\pi}{6}\right]$

h) $\int_0^{\sqrt{3}}\frac{x\,dx}{\sqrt{x^2+1}}$ [1] i) $\int_0^{5}\frac{x+2}{\sqrt{x+4}}dx$ $\left[\frac{26}{3}\right]$ j) $\int_0^{\frac{\pi}{2}}\frac{\cos x}{\sqrt{2+\sin x}}dx$ $\left[2\sqrt{3}-2\sqrt{2}\right]$ k) $\int_{\sqrt{2}}^{\sqrt{5}}\frac{x}{\sqrt{x^2-1}}dx$ $[1]$.

2. Obliczyć (lub wykazać rozbieżność) całki niewłaściwe: a) $\int_{-\infty}^{+\infty}\frac{2x\,dx}{x^2+1}$ [rozbieżna] b) $\int_0^{\infty}\frac{dx}{x^2-4x+8}$ $\left[\frac{3\pi}{4}\right]$

c) $\int_1^{\infty}\frac{4x+\sqrt{x}}{x^2}dx$ [rozbież.] d) $\int_1^{\infty}\frac{4x+\sqrt{x}}{x^3}dx$ $\left[\frac{10}{3}\right]$ e) $\int_{-1}^{0}\frac{e^{-\frac{1}{x}}}{x^3}dx$ [rozbież.] f) $\int_0^{\frac{1}{e}}\frac{dx}{x\ln^2x}$ [1] g) $\int_0^{4}\frac{4x}{x^2\sqrt{x}}dx$ [rozbież.]

h) $\int_0^{1}\frac{\sqrt{x}}{3x}dx$ $\left[\frac{2}{3}\right]$ i) $\int_0^{2}\frac{dx}{(x-1)^2}$ j) $\int_{-1}^{1}\frac{dx}{\sqrt[3]{x^5}}$ k) $\int_0^{2}\frac{\sqrt{x}}{4x^3}dx$ l) $\int_{-1}^{2}\frac{1}{x^2-2x-3}dx$ [i, j, k, l - całki rozbieżne].

3. Obliczyć pole obszaru ograniczonego krzywymi (wykonać rysunek): a) $y=x-x^2,\ y=x^2-1$ $\left[\frac{9}{8}\right]$

b) $y=x^2+x-2,\ y=-x+1$ $\left[\frac{32}{3}\right]$ c) $y=x^2-6x+10,\ y=6x-x^2$ $\left[23\frac{1}{3}\right]$ d) $y=\frac{2}{x},\ y=x-1,\ x=4$ $\left[4-\ln\frac{1}{4}\right]$

e) $y=2-x,\ y=\sqrt{x},\ x=0$ $\left[\frac{7}{6}\right]$ f) $y^2=2-x,\ y=-x$ $\left[4\frac{1}{2}\right]$ g) $y=\frac{x^2}{4},\ y=\frac{8}{x^2+4}$ $\left[2\pi-\frac{4}{3}\right]$.

Rozwiązania niektórych zadań.

1. e) $\int_0^{\frac{\pi}{3}}\operatorname{tg}^3x\,dx=\int_0^{\sqrt{3}}\frac{t^3}{1+t^2}dt=\int_0^{\sqrt{3}}\frac{t^3+t-t}{1+t^2}dt=\int_0^{\sqrt{3}}\frac{t^3+t}{1+t^2}dt-\frac{1}{2}\int_0^{\sqrt{3}}\frac{2t}{t^2+1}=\int_0^{\sqrt{3}}t\,dt-\frac{1}{2}\ln(t^2+1)\Big|_0^{\sqrt{3}}=$

x	0	$\frac{\pi}{3}$
t	0	$\sqrt{3}$

$=\frac{t^2}{2}\Big|_0^{\sqrt{3}}-\frac{1}{2}\ln 4-\frac{1}{2}\ln 1=\frac{3}{2}-\ln 2.$

f) $\int_0^{\sqrt{\frac{\pi}{2}}}x\sin x^2dx=\left[\begin{array}{c}x^2=t\\2xdx=dt\end{array}\right]=\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin t\,dt=-\frac{1}{2}\cos t\Big|_0^{\frac{\pi}{2}}=0+\frac{1}{2}\cdot 1=\frac{1}{2}.$

x	0	$\sqrt{\frac{\pi}{2}}$
t	0	$\frac{\pi}{2}$

g) $\int_0^{\frac{\pi}{3}}\cos^2x\,dx=\left[\frac{1+\cos 2x}{2}\right]=\frac{1}{2}\int_0^{\frac{\pi}{3}}(1+\cos 2x)dx=\frac{1}{2}x\Big|_0^{\frac{\pi}{3}}+\frac{1}{4}\sin 2x\Big|_0^{\frac{\pi}{3}}=\frac{\pi}{6}+\frac{1}{4}\sin\frac{2\pi}{3}=\frac{\pi}{6}+\frac{1}{4}\cdot\frac{\sqrt{3}}{2}=\frac{\pi}{6}+\frac{\sqrt{3}}{8}.$

h) $\int_0^{\sqrt{3}} \frac{xdx}{\sqrt{x^2+1}} = \left[\begin{array}{c} \sqrt{x^2+1}=t \\ x^2+1=t^2 \\ xdx = tdt \end{array}\right] = \int_1^2 \frac{t\,dt}{t} = t\ \Big|_1^2 = 2-1=1.$

j) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{2+\sin x}}dx = \left[\begin{array}{c} t=\sqrt{2+\sin x} \\ t^2 = 2+\sin x \\ 2tdt = \cos xdx \end{array}\right] = \int_{\sqrt{2}}^{\sqrt{3}} \frac{2tdt}{t} = 2t\Big|_{\sqrt{2}}^{\sqrt{3}} = 2\sqrt{3}-2\sqrt{2}.$

x	0	$\frac{\pi}{2}$
t	$\sqrt{2}$	$\sqrt{3}$

2.a) $\int_{-\infty}^{+\infty} \frac{2x\,dx}{x^2+1} = \int_{-\infty}^{0} \frac{2x\,dx}{x^2+1} + \int_0^{+\infty} \frac{2x\,dx}{x^2+1} = \lim_{a\to-\infty} \int_a^0 \frac{2x\,dx}{x^2+1} + \lim_{b\to+\infty} \int_0^b \frac{2x\,dx}{x^2+1} = \lim_{a\to-\infty} \ln(x^2+1) + \lim_{b\to+\infty} \ln(x^2+1) = \infty+\infty,$

więc całka rozbieżna.

c) $\int_1^{\infty} \frac{4x+\sqrt{x}}{x^2}dx = \lim_{a\to\infty} \int_1^a \frac{4x+\sqrt{x}}{x^2}dx = \lim_{a\to\infty} (4\ln x - \frac{2}{\sqrt{x}})\ \Big|_1^a = \lim_{a\to\infty} (4\ln a - \frac{2}{a} - 4\ln 1 + 2) = \infty$, więc całka rozbieżna.

d) $\int_1^{\infty} \frac{4x+\sqrt{x}}{x^3}dx = \lim_{a\to\infty} \int_1^a \frac{4x+\sqrt{x}}{x^3}dx = \lim_{a\to\infty} \left(-\frac{4}{x} - \frac{2}{3\sqrt{x^3}}\right)\Big|_1^a = \lim_{a\to\infty}\left(-\frac{4}{a} - \frac{2}{3\sqrt{a^3}} + 4 + \frac{2}{3}\right) = \frac{10}{3}.$

f) $\int_0^{\frac{1}{e}} \frac{dx}{x\ln^2 x} = \lim_{a\to 0^+} \left(-\frac{1}{\ln x}\right)\Big|_a^{\frac{1}{e}} = \lim_{a\to 0^+}\left(-\frac{1}{\ln e^{-1}} + \frac{1}{\ln a}\right) = -(-1)+0 = 1$

$\int \frac{dx}{x\ln^2 x} = \left[\begin{array}{c} \ln x = t \\ \frac{1}{x}dx = dt \end{array}\right] = \int \frac{dt}{t^2} = -\frac{1}{t} = -\frac{1}{\ln x}.$

i) $\int_0^2 \frac{dx}{(x-1)^2} = \int_0^1 \frac{dx}{(x-1)^2} + \int_1^2 \frac{dx}{(x-1)^2} = \lim_{a\to 1^-} \int_0^a \frac{dx}{(x-1)^2} + \lim_{a\to 1^+} \int_a^1 \frac{dx}{(x-1)^2} = \lim_{a\to 1^-}\left(-\frac{1}{x-1}\ \Big|_0^a\right) + \lim_{a\to 1^+}\left(-\frac{1}{x-1}\ \Big|_a^0\right) =$

$\lim_{a\to 1^-}\left(-\frac{1}{a-1} - 1\right) + \lim_{a\to 1^+}\left(1 + \frac{1}{a-1}\right) = \infty + \infty = \infty$ całka rozbieżna.

l) $\int_{-1}^2 \frac{1}{x^2-2x-3}dx = \lim_{a\to -1^+} (-\frac{1}{4}\ln|x+1| + \frac{1}{4}\ln|x-3|)\ \Big|_a^2 = \lim_{a\to -1^+}\left(-\frac{1}{4}\ln 3 + \frac{1}{4}\ln|2-3| + \frac{1}{4}\ln(a+1) - \frac{1}{4}\ln|a-3|\right) =$

$\frac{1}{4}\ln\left(\frac{4}{3}\right) - \infty = -\infty$ całka rozbieżna.

$\frac{1}{x^2-2x-3} = \frac{1}{(x+1)(x-3)} = \frac{-\frac{1}{4}}{x+1} + \frac{\frac{1}{4}}{x+3}$; $\int \frac{1}{x^2-2x-3}dx = -\frac{1}{4}\ln|x+1| + \frac{1}{4}\ln|x-3|.$

3 f) $y^2 = 2-x,\ y=-x$;

$P = \int_{-2}^1 (\sqrt{2-x}+x)dx + \int_1^2 (\sqrt{2-x}+\sqrt{2-x})dx = -\frac{2\sqrt{(2-x)^3}}{3}\ \Big|_{-2}^1 + \frac{x^2}{2}\ \Big|_{-2}^1 - \frac{4\sqrt{(2-x)^3}}{3}\ \Big|_1^2$

$= -\frac{2}{3} + \frac{16}{3} + \frac{1}{2} - 2 - 0 + \frac{4}{3} = 6 - 2 + \frac{1}{2} = \frac{9}{2}.$

3 g) $y = \frac{x^2}{4},\ y = \frac{8}{x^2+4}$ punkty wspólne dla $x=-2$ lub $x=2$.

$P = \int_{-2}^2 \left(\frac{8}{x^2+4} - \frac{x^2}{4}\right)dx = 4\mathrm{arctg}\left(\frac{x}{2}\right)\ \Big|_{-2}^2 - \frac{x^3}{12}\ \Big|_{-2}^2 = 4\mathrm{arctg}1 - 4\mathrm{arctg}(-1) - \frac{2}{3} - \frac{2}{3} = 2\pi - \frac{4}{3}.$