# Równania różniczkowe cząstkowe drugiego rzędu

**Zadanie 1.** Wyznacz obszary w których zachowuje się typ równania:

$a)\ x\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

$b)\ x\frac{\partial^2 u}{\partial x^2}-y^2\frac{\partial^2 u}{\partial y^2}=0$

$c)\ y\frac{\partial^2 u}{\partial x^2}+x^2\frac{\partial^2 u}{\partial y^2}=0$

$d)\ (x^2+y^{2)}\frac{\partial^2 u}{\partial x^2}-2\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}-2\frac{\partial u}{\partial x}=0$

Odpowiedzi

a)

typ eliptyczny – półpłaszczyzna $x>0$
typ hiperboliczny - półpłaszczyzna $x<0$
typ paraboliczny: ( $x=0$ )

b)

typ eliptyczny: ( $x<0\wedge y\neq 0$ )
typ hiperboliczny: ( $x>0\wedge y\neq 0$ )
typ paraboliczny: ( $x=0\ oraz\ y=0$ )

c)

typ eliptyczny: [ $(x>0\wedge y>0)\vee(x<0\wedge y<0)$ ]
typ hiperboliczny: [ $(x>0\wedge y<0)\vee(x<0\wedge y>0)$ ]
typ paraboliczny: ( $x=0\wedge y=0$ )

d)

typ hiperboliczny: ( $x^2+y^2<1$ )
typ paraboliczny: ( $x^2+y^2=1$ )
typ eliptyczny: ( $x^2+y^2>1$ )

**Zadanie 2.** Wyznacz charakterystyki równania:

a) $\frac{\partial^2 u}{\partial x^2}+6\frac{\partial^2 u}{\partial x\partial y}+5\frac{\partial^2 u}{\partial y^2}=0$

b) $\frac{\partial^2 u}{\partial x^2}+6\frac{\partial^2 u}{\partial x\partial y}+10\frac{\partial^2 u}{\partial y^2}=0$

c) $\frac{\partial^2 u}{\partial x^2}-4\frac{\partial^2 u}{\partial x\partial y}+4\frac{\partial^2 u}{\partial y^2}=0$

d) $x^2\frac{\partial^2 u}{\partial x^2}-y^2\frac{\partial^2 u}{\partial y^2}=0$

Odpowiedzi

$a)\ \mathrm{y - 5x = C_1;\ \ y - x = C_2}$
$b)\ \mathrm{y - (-3 - 4i)x = C_1;\ \ - y(3 + 4i)x = C_2}$
$c)\ \mathrm{y = -2x + C_1}$
$d)\ \frac{\mathrm{y}}{\mathrm{x}} = C_1;\ \ \mathrm{xy = C_2}$

**Zadanie 3.** Sprowadź do postaci kanonicznej równania:

$$a)\frac{\partial^2 u}{\partial x^2} + 6\frac{\partial^2 u}{\partial x \partial y} + 5\frac{\partial^2 u}{\partial y^2} = 0$$
$$b)\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 8\frac{\partial^2 u}{\partial y^2} = 0$$
$$c)\frac{\partial^2 u}{\partial x^2} + 6\frac{\partial^2 u}{\partial x \partial y} + 9\frac{\partial^2 u}{\partial y^2} + u = 0$$
$$d)\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 0$$
$$e)\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} + 6\frac{\partial u}{\partial y} = 0$$
$$f)\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 5\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$$

Odpowiedzi

$$a)\ \ \xi = \mathrm{y - 6x},\ \ \eta = \mathrm{-4x};\ \ \frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2}$$
$$b)\ \ \xi = \mathrm{2y + 4x},\ \ \eta = \mathrm{4x}\ \ \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = 0$$
$$c)\ \ \xi = \mathrm{y - 3x},\ \ \eta = \mathrm{x};\ \ \frac{\partial^2 u}{\partial \eta^2} + u = 0$$
$$d)\ \ \xi = \mathrm{y - x},\ \ \eta = \mathrm{x};\ \ \frac{\partial^2 u}{\partial u^2} + \frac{\partial u}{\partial \xi} = 0$$
$$e)\ \ \xi = \mathrm{x + y},\ \ \eta = \mathrm{3x - y};\ \ \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{1}{2}\frac{\partial u}{\partial \xi} = 0$$
$$f)\ \ \xi = \mathrm{2x - y},\ \ \eta = \mathrm{x};\ \ \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \frac{\partial u}{\partial \eta} = 0$$

**Zadanie 4** Sprowadź do postaci kanonicznej równania:

a) $tg^2 x \frac{\partial^2 u}{\partial x^2} - 2ytgx \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + tg^3 x \frac{\partial u}{\partial x} = 0$

b) $y \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

c) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} - 3y^2 \frac{\partial^2 u}{\partial y^2} - 2x \frac{\partial u}{\partial x} + 4y \frac{\partial u}{\partial y} + 16x^4 u = 0$

d) $\frac{\partial^2 u}{\partial x^2} - 2\cos x \frac{\partial^2 u}{\partial x \partial y} - (3 + \sin^2 x) \frac{\partial^2 u}{\partial y^2} = 0$

e) $\frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + 5 \frac{\partial^2 u}{\partial y^2} = 0$

Odpowiedzi :

a) $\xi = y\sin x, \quad \eta = y; \qquad \frac{\partial^2 u}{\partial \eta^2} - \frac{2\xi}{\eta^2} \cdot \frac{\partial u}{\partial \xi} = 0$

b) $\xi = x, \qquad \eta = \frac{2}{3} y^{\frac{3}{2}}; \qquad \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \frac{1}{3\eta} \cdot \frac{\partial u}{\partial \eta} = 0$

c) $\xi = xy, \quad \eta = \frac{x^3}{y}; \qquad \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{1}{4\eta} \frac{\partial u}{\partial \xi} - \frac{1}{\xi} \cdot \frac{\partial u}{\partial \eta} + u = 0$

d) $12 \frac{\partial^2 u}{\partial \xi \partial \eta} + \sin \frac{\eta - \xi}{4} (\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}) = 0$

e) $\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = 0$

**Zadanie 5** Wyznacz ogólne rozwiązanie następujących równań:

a) $\frac{\partial^2 u}{\partial x \partial y} - 2y \frac{\partial u}{\partial x} = 0$

b) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} = 0$

c) $\frac{\partial^2 u}{\partial x \partial y} + 4x^3 = 0$

d) $\frac{\partial^2 u}{\partial x^2} - 2\sin x \frac{\partial^2 u}{\partial x \partial y} - \cos^2 x \frac{\partial^2 u}{\partial y^2} - \cos x \frac{\partial u}{\partial y} = 0$

e) $x^2 \frac{\partial^2 u}{\partial x^2} - y^2 \frac{\partial^2 u}{\partial y^2} - 2y \frac{\partial u}{\partial y} = 0$

f) $\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} = 0$

g) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} = 0$

h) $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} - y = 0$

Odpowiedzi:

a) $u(x, y) = A(x)e^{y^2} + \int_0^y \varphi(t)e^{y^2-t^2}dt$

b) $u(x, y) = A(y) + B(x - y)$

c) $u(x, y) = -x^4 y + f(x) + g(y)$

d) $u(x, y) = \varphi(x + y - \cos x) + \psi(x - y + \cos x)$

e) $u(x, y) = \sqrt{\frac{x}{y}}\varphi(x, y) + \psi(\frac{y}{x})$

f) $u(x, y) = A(y)e^{-x} + F(x) \qquad F(x) = \int_0^x \varphi(t)e^{t-x}dt$

g) $u(x, y) = F(y) + G(x - y)$

h) $u(x, y) = A(y)e^{x} - (x+1)y + \varphi(y)$

**Zadanie 6** Wyznacz rozwiązanie równania spełniające dane warunki:

a) $\frac{\partial^2 u}{\partial x \partial y} - 2x\sin y = 0 \qquad u(x,0) = x^2 \quad u(0, y) = \sin y$

b) $\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad u(x,0) = x^5 \quad u(0, y) = y^3$

c) $\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial y} = 0 \qquad u(x,0) = x^3 \quad u(0, y) = y^7$

d) $\frac{\partial^2 u}{\partial x \partial y} = 1 \qquad u(x,0) = x^5 \quad u(0, y) = y^2$

e) $\frac{\partial^2 u}{\partial x \partial y} - 3x^2\frac{\partial u}{\partial y} = 0 \qquad u(x,0) = 5x^4 + x^2 \quad u(0, y) = 3y^3$

*Odpowiedzi* :

*a*) $u(x, y) = x^2(2 - \cos y) + \sin y$

*b*) $u(x, y) = x^5 + x^3 + (y - x)^3$

*c*) $u(x, y) = e^x y^7 + x^3$

*d*) $u(x, y) = x^5 + xy + y^2$

*e*) $u(x, y) = 5x^4 + x^2 + 3y^3 e^{x^3}$

**Zadanie 7** Znajdź rozwiązanie równania spełniające podane warunki:

a) $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial^2 u}{\partial y^2} = 0 \qquad u(x,0) = 3x^2 \qquad \frac{\partial u(x,0)}{\partial y} = 0$

b) $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial^2 u}{\partial y^2} = 0 \qquad u(x,x) = 0 \qquad \frac{\partial u(x,x)}{\partial y} = 3x^2$

c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - 2\frac{\partial^2 u}{\partial y^2} = 0 \qquad u(0, y) = y^5 - 5y \qquad \frac{\partial u}{\partial x}(0, y) = -2y - 10$

d) $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial^2 u}{\partial y^2} = 0 \qquad u(x,-x) = 0 \qquad \frac{\partial u}{\partial y}(x,-x) = 4x^3$

Odpowiedzi:

$a)\ u(x, y) = 3x^2 + y^2$

$b)\ u(x, y) = \frac{1}{8}\left[(y + x)^3 + (y - 3x)^3\right]$

$c)\ u(x, y) = x^2 - 2xy + y^2 - 10x - 5y$

$d)\ u(x, y) = \frac{1}{16}\left[(y + 3x)^4 - (y - x)^4\right]$